Nonparametric estimation of a shot-noise process

Paul Ilhe, paul.ilhe@telecom-paristech.fr

Joint work with François Roueff, Éric Moulines & Antoine Souloumiac

Journée TSI, 2016.
Outline

Introduction
- Shot-noise processes on the real line
- $\gamma$-spectroscopy

Statistical estimation
- Assumptions
- Estimator
- RKHS
- B-spline method

Numerical results
Outline

Introduction

Shot-noise processes on the real line
\( \gamma \)-spectroscopy

Statistical estimation

Numerical results
A stationary shot-noise process is a stochastic process $\mathbf{X} = (X_t)_{t \in \mathbb{R}}$ that can be written

$$X_t := \sum_{k : T_k \leq t} Y_k h(t - T_k)$$

where $N := \sum_{k \in \mathbb{Z}} \delta(T_k, Y_k)$ is a Poisson random measure such that

- the sequence of time events $(T_k)_{k}$ follows a Poisson point process on $\mathbb{R}$ with intensity $\lambda$,
- $(Y_k)_{k}$ are independent of $(T_k)_{k}$ and mutually i.i.d. with density $f$, and $h$ is a measurable real-valued function.
A stationary shot-noise process is a stochastic process $X = (X_t)_{t \in \mathbb{R}}$ that can be written

$$X_t := \sum_{k : T_k \leq t} Y_k h(t - T_k)$$

where $N := \sum_{k \in \mathbb{Z}} \delta(T_k, Y_k)$ is a Poisson random measure such that

- the sequence of time events $(T_k)_k$ follows a Poisson point process on $\mathbb{R}$ with intensity $\lambda$,
- $(Y_k)_k$ are independent of $(T_k)_k$ and mutually i.i.d. with density $f$. 

\[\text{Definition}\] 

**Introduction**

Statistical estimation 
Numerical results 

**Shot-noise processes on the real line** 

$\gamma$-spectroscopy 

- Shot-noise processes: definition
**Shot-noise process: definition**

**Definition**

A stationary shot-noise process is a stochastic process \( X = (X_t)_{t \in \mathbb{R}} \) that can be written

\[
X_t := \sum_{k : T_k \leq t} Y_k h(t - T_k)
\]

where \( N := \sum_{k \in \mathbb{Z}} \delta(T_k, Y_k) \) is a Poisson random measure such that

- the sequence of time events \((T_k)_k\) follows a Poisson point process on \(\mathbb{R}\) with intensity \(\lambda\),

- \((Y_k)_k\) are independent of \((T_k)_k\) and mutually i.i.d. with density \(f\),

and \(h\) is a measurable real-valued function.
The elements of a shot-noise

Properties

- A shot-noise is fully characterized by its triplet \((\lambda, f, h)\).
- The process is well defined as soon as
  \[
  \lambda \int_{\mathbb{R}} \int_{\mathbb{R}} |y h(s)| f(y) dy ds < \infty ,
  \]
- The characteristic function \(\varphi\) of \(X_0\) is given for \(u \in \mathbb{R}\) by
  \[
  (2) \quad \varphi_{X_0}(u) = \mathbb{E} \left[ e^{i u X_0} \right] = \exp \left( \lambda \int_{\mathbb{R}} [\varphi_Y(u h(s)) - 1] ds \right).
  \]
Introduction
Statistical estimation
Numerical results

Shot-noise processes on the real line
γ-spectroscopy

Illustration

Figure: Sample path of a shot-noise process.

Vocabulary

Intensity: $\lambda$, Marks’ density: $f$, Impulse response: $h$. 
\(\gamma\)-spectroscopy

**Interpretation**

- \((T_k)_k\): interaction photon-matter events,
- \((Y_k)_k\): energies recorded by the detector,
- \(h\): electrical current generated by the detector.

**Inputs**

A low frequency sample of the shot-noise \(X_{\delta}, \ldots, X_{n\delta}\)

**Objectives**

- Estimate the mean number of photons in one second: \(\lambda\),
- Estimate the density \(f\) of the marks \((Y_k)_k\).
\textbf{\(\gamma\)-spectroscopy: example 1}

- Source: Americium 241
- Intensity: \(5 \times 10^5\) cps

Example of a temporal signal in \(\gamma\)-spectroscopy.
**γ-spectroscopy: example 2**

- **Source:** Americium 241
- **Intensity:** $4 \cdot 10^6$ cps

Example of a temporal signal in γ-spectroscopy.
Outline

Introduction

Statistical estimation
  Assumptions
  Estimator
  RKHS
  B-spline method

Numerical results
Shot-noise process: statistical problem

Problem and assumptions

Based on

- A low frequency sample \( X_\delta, \ldots, X_{n\delta} \),
- A complete knowledge of the impulse response \( h \),

we aim to estimate the flow function

\[
f_\lambda := \lambda f .
\]
Assumptions

- the mark’s density $f$ belongs to

$$(H-1) \mathcal{W}^2 := \{ f : [0, 1] \rightarrow \mathbb{R}_+ : f, f' \text{ absolutely continuous,}$$

$$f(1) = f'(1) = 0 \text{ and } \int_0^1 |f''(t)|^2 \, dt < \infty \}. \,$$

- There exists two positive constants $C_0$ and $\alpha$ s.t.

$$(H-2) \quad |h(t)| \leq C_0 e^{-\alpha t}, \quad t \in \mathbb{R}.$$
Estimation problem (1/3)

Main tool

If $\mathbb{E}[|Y_0|] < \infty$, we have for every $u \in \mathbb{R}$

$$g(u) := \frac{\varphi'(u)}{\varphi(u)} = \int_0^1 \int_{\mathbb{R}} i x h(s) e^{iuxh(s)} \lambda f(x) dx ds =: K_h[f_\lambda](u).$$

Functional inverse problem

We transformed the estimation problem into a linear inverse problem with

- $K_h$ a BLE acting from $\mathcal{W}^2$ to another Hilbert space,
- we can construct an empirical estimator $\hat{g}_n$ of $g$. 
Estimation problem (2/3)

Plug-in estimator

\[ \hat{g}_n(u) := \frac{\phi'_n(u)}{\phi_n(u)} = \frac{\sum_{k=1}^{n} iX_k e^{iuX_k}}{\sum_{k=1}^{n} e^{iuX_k}} , \quad u \in \mathbb{R}. \]

error terms:  
\[ \epsilon_n(u) = \sqrt{n} (\hat{g}_n(u) - g(u)) , \quad u \in \mathbb{R}. \]

Theorem

Under Assumption (H-2), we have for all positive \( K \) that

\[ \epsilon_n \Rightarrow Z \quad \text{in} \quad C([-K, K], \mathbb{C}) , \]

where \( Z \) is a Gaussian process with covariance matrix \( \Omega \).
Estimation problem (3/3)

**Discrete sampling**

\( \hat{g}_n \) and \( g \) are evaluated on a discrete grid \( u = \{u_1, \ldots, u_N\} \).

- \( \hat{g}_n(u) = [\hat{g}_n(u_1), \ldots, \hat{g}_n(u_N)]^T \),
- \( K_u[\xi] = [K_h[\xi](u_1), \ldots, K_h[\xi](u_N)]^T \).

**Minimization problem**

Given a positive definite matrix \( W \) of size \( N \) and \( \mu \) a positive number, we define

\[
(3) \quad \hat{f}_{n,\lambda} := \arg\min_{\xi \in \mathcal{W}_2^+} \left\| W^{1/2} \cdot \{ \hat{g}_n(u) - K_u[\xi] \} \right\|_2^2 + \mu \| \xi \|_{\mathcal{W}_2}^2.
\]
Reproducing Kernel

The functional space \( \mathcal{W}^2 \) is a RKHS with r.k.

\[
R_x(y) := R(x, y) := \int_0^1 (u - x)_+ (u - y)_+ du , \quad x, y \in [0, 1].
\]

Corollary

The solution of (3) can be rewritten \( \hat{f}_{n, \lambda} = \sum_{k=1}^N \hat{\alpha}_{n,k} \eta_i \) with

- \( \eta_i : x \rightarrow K_h[R_x(\cdot)](u_i) \) for \( i = 1, \ldots, N \).
- The coefficients \( \hat{\alpha}_n \) are solution of

\[
\arg\min_{\alpha \in \mathbb{C}^N} \| \mathcal{W}^{1/2} \cdot \{ \hat{g}_n(u) - G\alpha \} \|_2^2 + \mu \alpha^T G \alpha .
\]

with \( G \) the Gram matrix associated to \( \eta_1, \ldots, \eta_N \).
We approximate $\mathcal{W}^2$ by B-splines spaces $(S^3_k)_k$ of order 3.

- the approximation error is in $O(k^{-2})$,
- the functions are locally supported,
- the penalty term $\|\xi\|^2_{\mathcal{W}^2}$ is easily expressible.

**Figure**: B-splines basis of order 3. Left: $k = 10$. Right: $k = 20$. 
Practical procedure

- Choice of $N = k = 2^\lceil \log(n) \rceil$,
- Choice of $W = \hat{\Omega}_n^{-1}$ with $\hat{\Omega}_n$ a positive definite estimator of $\Omega$,
- Selection of the regularizing parameter $\mu$ by generalized cross-validation
Outline

Introduction

Statistical estimation

Numerical results
Simulated shot-noise

- $\lambda = 3$, $\delta = 0.01$,
- $h : t \rightarrow t \ e^{-10t} \mathbb{1}_{\{t\geq 0\}}$
- $f = 0.7 \mathcal{N}(0.3; 2.5 \times 10^{-3}) + 0.3 \mathcal{N}(0.6; 2.5 \times 10^{-3})$

Simulated shot-noise and associated sample points.
RKHS

Mitigated results:

- Retrieves the modes of the distribution
- Positivity constraints hard to handle
- Basis function not adapted

**Figure:** Density estimation using the representer’s theorem.
B-splines

- $k = N = 2^\lfloor \log(n) \rfloor$, 
- Computation of $\hat{\Omega}_n$ by bootstrap, 
- Choice of the penalization $\mu$ by generalized cross-validation.
Nuclear dataset

- source: mixture of Am 241 and Pu 238,
- intensity: $5 \times 10^5$ cps.